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IDEALS IN GROUP RINGS OF FREE PRODUCTS

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ABSTRACT
Let G be a group and RG be its group ring. If A is a nonzero ideal of RG, we prove that for certain normal subgroups H of G, including all nontrivial subgroups of G when G is a free product, $A \cap RH \neq 0$.

Suppose R is a ring (associative with unit), G is a group, and N is a subgroup of G. An intersection theorem for the group ring RG is a theorem which asserts, under suitable hypotheses, that every nonzero ideal of RG has a nonzero intersection with RN. The most notable intersection theorem is that of Zalesski for solvable groups [1, p. 364]. It plays a crucial role in the solution of the semisimplicity problem for group rings of solvable groups by Hampton, Passman and Zalesski [cf. 1, p. 332]. Several other examples and applications of intersection theorems can be found in [1].

In this paper we prove an intersection theorem for certain normal subgroups of G, including all nontrivial normal subgroups of G when G is a free product. Before stating these theorems we introduce the following notation.

Let R be a ring, G a group, N a subgroup of G, and $g \in G$.

$$C_N(g) = \text{centralizer of } g \text{ in } N;$$

$$D_N(G) = \{g \in G \mid |N : C_N(g)| < \infty\};$$

$$\Delta(G) = C_G(N) = \{g \in G \mid |G : C_G(g)| < \infty\}.$$

If $\alpha = a_1g_1 + \cdots + a_ng_n \in RG$, where $g_1, \cdots, g_n$ are distinct elements of G and $a_1, \cdots, a_n$ are nonzero elements of R, then the support of α, $\operatorname{supp}(\alpha)$, is the set $\{g_1, \cdots, g_n\}$.

Our main results are the following two theorems.

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THEOREM 1. *Let R be a domain (not necessarily commutative), G a group, and N a normal subgroup of G. Suppose that $D_N(G)=1$. Then $A \cap RN \neq 0$ whenever A is a nonzero ideal of RG.*

We remark that if N is a normal subgroup of G, then $D_N(G)=1$ if and only if $C_N(G)=1$ and $\Delta(N)=1$.

THEOREM 2. *Let R be a domain (not necessarily commutative), $G=H*K$ a nontrivial free product, and $N\neq 1$ a normal subgroup of G. Then $A \cap RN \neq 0$ whenever A is a nonzero ideal of RG.*

The proof of Theorem 1 is based on the next lemma, which is essentially the same as lemma 2 of [2]. However, we include a proof.

LEMMA 3. *Let R be a domain, G a group, N a subgroup of G, and α, β nonzero elements of RG with* $\mathrm{supp}(\alpha)\neq \mathrm{supp}(\beta)$. *Suppose $\alpha x\beta=\beta x\alpha$ for all $x \in N$. Then $D_N(G)\neq 1$.*

PROOF. Let $\mathrm{supp}(\alpha)=\{g_1,\cdots,g_n\}$, $\mathrm{supp}(\beta)=\{h_1,\cdots,h_n\}$. By rearranging indices and interchanging α and β if necessary, we may assume that $g_1 \notin \mathrm{supp}(\beta)$.

Whenever $g_i^{-1}g_1$ and $h_jh_1^{-1}$ $(i,j\neq 1)$ are conjugate by an element of N, choose $u_{ij}\in N$ such that

$$u_{ij}^{-1}(g_i^{-1}g_1)u_{ij}=h_jh_1^{-1};$$

and whenever $h_i^{-1}g_1$ and $g_jh_1^{-1}$ are conjugate by an element of N, choose $v_{ij}\in N$ such that

$$v_{ij}^{-1}(h_i^{-1}g_1)v_{ij}=g_ih_1^{-1}.$$

Now suppose $x \in N$ and consider

$$\alpha x\beta-\beta x\alpha=0.$$

Since $g_1\in \mathrm{supp}(\alpha)$, $h_1 \in \mathrm{supp}(\beta)$, a cancellation of the product g_1xh_1 must occur in the preceding equation (here the hypothesis that R is a domain is used implicitly). This means that either

(1) $g_1xh_1=g_ixh_j$, where $i,j\neq 1$, or

(2) $g_1xh_1=h_ixg_j$

for some i and j. If (1) occurs, then $x\in C_N(g_i^{-1}g_1)u_{ij}$ and if (2) occurs, then $x \in C_N(h_i^{-1}g_1)v_{ij}$. Since this is true for any $x\in N$, N is a union of finitely many right cosets of the finitely many subgroups $C_N(g_i^{-1}g_1)$, $(i\neq 1)$, $C_N(h_i^{-1}g_1)$.

This implies [1, p. 120] that one of these subgroups has finite index in N, and hence that either some $g_i^{-1}g_1$ $(i\neq 1)$ or some $h_i^{-1}g_1$ lies in $D_N(G)$. Since

$g_1 \notin \operatorname{supp}(\beta) = \{h_1, \cdots, h_n\}$, these elements are all nontrivial. Hence $D_N(G) \neq 1$. ■

Proof of Theorem 1. Let $\{g_i\}$ be a set of coset representatives of N in G with $1 \in \{g_i\}$. Then every nonzero element of RG has a unique expression of the form

$$u = u_1 g_1 + \cdots + u_n g_n,$$

where $u_1, \cdots, u_n$ are nonzero elements of RN and $g_1, \cdots, g_n \in \{g_i\}$. Call $n \geqq 1$ the length of u, and let u be a nonzero element of A whose length is minimal. By right multiplying u by g_1^{-1} it may be assumed that $g_1 = 1$.

We claim that u has length 1. For suppose conversely that

$$u = u_1 + u_2 g_2 + \cdots + u_n g_n$$

has length $n \geqq 2$. Note that $\operatorname{supp}(u_1) \neq \operatorname{supp}(u_2 g_2)$ since $g_1 = 1$ and g_2 represent distinct cosets of N in G. Since $D_N(G) = 1$, the lemma asserts that there is an $x \in N$ such that $u_1 x u_2 g_2 \neq u_2 g_2 x u_1$. Consider

$$\begin{aligned} u_1 x u - u x u_1 &= (u_1 x u_2 g_2 - u_2 g_2 x u_1) + \cdots + (u_1 x u_n g_n - u_n g_n x u_1) \\ &= (u_1 x u_2 - u_2 g_2 x u_1 g_2^{-1}) g_2 + \cdots + (u_1 x u_n - u_n g_n x u_1 g_n^{-1}) g_n. \end{aligned}$$

This element lies in A, is nonzero (since $u_1 x u_2 \neq u_2 g_2 x u_1 g_2^{-1}$) and has length $< n$, contradicting the minimal choice of u.

Thus u has length 1, so u is a nonzero element of $A \cap RN$. ■

Proof of Theorem 2. We have to consider two cases separately, $G \neq \mathbf{Z}_2 * \mathbf{Z}_2$ and $G = \mathbf{Z}_2 * \mathbf{Z}_2$. Note that if $G \neq \mathbf{Z}_2 * \mathbf{Z}_2$, then $\Delta(G) = 1$ and no normal subgroup of G is isomorphic to $\mathbf{Z}$.

Case I. $G \neq \mathbf{Z}_2 * \mathbf{Z}_2$. We claim that $D_N(G) = 1$. For suppose that $x \in D_N(G)$ and let $L = \langle N, x \rangle$. L cannot be conjugate to a subgroup of a free factor of G since it contains N, a normal subgroup of G, and $L \neq \mathbf{Z}$ or $\mathbf{Z}_2 * \mathbf{Z}_2$ since either possibility implies that G has a normal subgroup isomorphic to $\mathbf{Z}$. Hence by the Kurosh subgroup theorem L is a nontrivial free product. Since $L \neq \mathbf{Z}_2 * \mathbf{Z}_2$, $\Delta(L) = 1$. Thus $x = 1$ since $x \in \Delta(L)$, and so $D_N(G) = 1$.

Applying Theorem 1 then gives the desired result.

Case II. $G = \mathbf{Z}_2 * \mathbf{Z}_2$. Then G is infinite dihedral and has a presentation

$$G = \langle g, h \mid hgh^{-1} = g^{-1}, h^2 = 1 \rangle.$$

Since N is a normal subgroup of G, $N \geqq \langle g^t \rangle$ for some positive integer t. We show that $A \cap RN \neq 0$ by a series of reductions:

(1) $A \cap SG \neq 0$, where S is a commutative subring of R.

(2) $A \cap S\langle g \rangle \neq 0$.

(3) $A \cap S\langle g^t \rangle \neq 0$.

First, choose a nonzero element

$$\alpha = a_1 g_1 + \cdots + a_n g_n \qquad (a_i \in R,\ g_i \in G)$$

of A whose support is minimal. If some a_i and a_j do not commute, then

$$\alpha a_i - a_i \alpha = (a_1 a_i - a_i a_1) g_1 + \cdots + (a_n a_i - a_i a_n) g_n$$

is a nonzero element of A whose support is properly smaller than $\mathrm{supp}(\alpha)$. Hence all the a_i commute, and we can restrict ourselves to the commutative subring S (with unit) of R generated by $a_1, \cdots, a_n$.

Since $\{1, h\}$ is a set of coset representatives for $\langle g \rangle$ in G, α can be written

$$\alpha = \beta + \gamma h,$$

where $\beta, \gamma \in S\langle g \rangle$. If $\beta = 0$, then αh is a nonzero element of $A \cap S\langle g \rangle$ and if $\gamma = 0$, then α is a nonzero element of $A \cap S\langle g \rangle$. If $\beta, \gamma \neq 0$, note that either $h\beta h^{-1} \neq \beta$ or $h(g\beta)h^{-1} \neq g\beta$. In any case, by replacing α by $g\alpha$ if necessary we may suppose that $\beta \neq h\beta h^{-1}$. Then

$$\begin{aligned} \alpha\gamma - \gamma h \alpha h^{-1} &= (\beta + \gamma h)\gamma - \gamma h(\beta + \gamma h)h^{-1} \\ &= (\beta - h\beta h^{-1})\gamma \end{aligned}$$

is a nonzero element of $S\langle g \rangle$. Thus $A \cap S\langle g \rangle \neq 0$.

Finally, we need only observe that any nonzero ideal of $S\langle g \rangle$ has a nonzero intersection with $S\langle g^t \rangle$ ($t \geqq 1$), because $S\langle g \rangle$ is a commutative domain integral over $S\langle g^t \rangle$. This is an elementary fact from commutative algebra. ■

References

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